Definition:

- log pair (X, D) — X: normal variety, D: effective $\mathbb{R}$-divisor
- $a(E, X, D)$: discrepancy of E with respect to $K_X + D$
- $\operatorname{discr}(X, D) = \inf_E \{a(E, X, D) \mid \operatorname{codim} \operatorname{Center}_X(E) \geq 2\}$
- $\operatorname{totaldiscr}(X, D) = \inf_E \{a(E, X, D) \mid \operatorname{codim} \operatorname{Center}_X(E) \geq 1\}$
- log pair (X, B) is said to be ε-log terminal if $\operatorname{totaldiscr}(X, B) > -1+\varepsilon$
  ε-log canonical if $\operatorname{totaldiscr}(X, B) \geq -1+\varepsilon$

- (X, B) be a log pair of global type (X is projective)

  Log Fano variety (LF) if $K+B$ is lc and $-(K+B)$ is ample
  weak log Fano (WLF) if $K+B$ is lc and $-(K+B)$ is nef and big
  log semi-Fano (ls-Fano) if $K+B$ is lc and $-(K+B)$ is nef
  0-log pair if $K+B$ is lc and numerically trivial ($\pi_1(X)=0$ or $q(X)=0$)
  also called a log Calabi-Yau variety

- X normal projective variety.
  X is Fano type (FT) if it satisfies the following equivalent conditions,
  (i) $\exists$ $\mathbb{Q}$-boundary Δ s.t. (X, Δ) is a klt log Fano
  (ii) $\exists$ $\mathbb{Q}$-boundary Δ s.t. (X, Δ) is a klt weak log Fano
  (iii) $\exists$ $\mathbb{Q}$-boundary Δ s.t (X, Δ) is a klt 0-pair and the components of Δ generate $N^1(X)$
  (iv) For any divisor D, $\exists$ a $\mathbb{Q}$-boundary Δ s.t (X, Δ) is a klt 0-pair and $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$

- X normal variety, D is $\mathbb{R}$-divisor
  $\mathbb{Q}$-complement of K_X+D is a log divisor K_X+D' s.t. $D' \geq D$, K_X+D' is lc and $n(K_X+D') \sim 0$ for some $n \in \mathbb{Z}_{>0}$

  let $D = S + B$ (no common components), $S \geq 0$ effective integral, $\lfloor B \rfloor \leq 0$
  n-complement of K_X+D is $K_X + D^+$ ($\mathbb{Q}$-divisor) s.t
  (i) $n(K_X + D^+) \sim 0$ (in particular, nD^+ is an integral divisor)
  (ii) $K_X + D^+$ is lc
  (iii) $nD^+ \geq nS + \lfloor (n+1)B \rfloor$
  Note that: an n-complement is not necessarily a $\mathbb{Q}$-complement

- Alexeev's Borisov's
  Conjecture 1
  Fix a real number $\varepsilon > 0$.
  $(X, B = \sum b_i B_i)$ d-dimensional, lc, $-(K_X+B)$ nef (semi-Fano variety)
  Assume (i) $K+B$ is ε-lt
  (ii) X Fano type (FT) ($\exists$ some boundary Δ s.t (X, Δ) is a klt log Fano variety)

  Then X belongs to an algebraic family i.e. X is bounded in the moduli space

  Toric case was proved

- Conjecture 2: Log Canonical Adjunction (7.13.1)
  + Particular Case of Effective log Abundance Conjecture (7.13.2)
  + Effective Ajunction (7.13.3)

Notations, For a subset $\mathcal{R} \subset \mathbb{R}_{\geq 0}$. $\Phi(\mathcal{R}) := \{1 - \frac{r}{m} \mid m \in \mathbb{Z}, m>0, r \in \mathcal{R}\} \cap [0,1]$
If $\mathcal{R} = \{0, 1\}$, $\Phi(\mathcal{R})$ is the set of standard multiplicities
Assume $\mathcal{R}$ is rational and finite, denote $I(\mathcal{R}) := \operatorname{lcm}(\text{denominators of } r \in \mathcal{R} \setminus \{0\})$.

B has hyperstandard multiplicities with respect to $\mathcal{R}$ if $B \in \Phi(\mathcal{R})$

Def: a class of (relative) log pairs $(\mathcal{X}/\mathcal{Z} \ni 0, \mathcal{B})$, a point on each $Z \in \mathcal{Z}$

This class has bounded complements if $\exists$ a constant C s.t. for any log pair $(X/Z, B) \in (\mathcal{X}/\mathcal{Z}, \mathcal{B})$, the log divisor $K+B$ is n-complement near the fiber over 0 for some $n \leq C$

Main Theorem 1
Fix a finite subset $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$.
(X, B) klt log semi-Fano variety (i.e. $K+B$ is lc and $-(K+B)$ is nef)
$\dim = d$, FT, multiplicities of $B \in \Phi(\mathcal{R})$ (i.e. B has hyperstandard multiplicities with respect to $\mathcal{R}$)

Assume: LMMP in $\dim \leq d$
Conjecture 1 and Conjecture 2 hold in $\dim \leq d$
Then: $K+B$ has bounded complements: $\exists$ positive integer $n = n(d, \mathcal{R})$ divisible by denominators of all $r \in \mathcal{R}$, s.t. $K+B$ is n-complemented.
Moreover, $K+B$ is nI-complemented for $\forall I \in \mathbb{Z}_{>0}$

In particular $|-nK - nS - \lfloor (n+1) D \rfloor| \neq \emptyset$ $S := \lfloor B \rfloor$ $D = B - S$
If $B = 0$ $|-nK| \neq \emptyset$ and n depends only on d.

Main Theorem 2
Fix a finite subset $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$
(X, B) be a 0-pair (log CY variety)
$\dim = d$ FT multiplicities of $B \in \Phi(\mathcal{R})$
Assume: LMMP in $\dim \leq d$
Conjecture 1 and Conjecture 2 hold in $\dim \leq d$
Then $\exists$ a positive integer $n = n(d, \mathcal{R})$ s.t $n(K_X+B) \sim 0$

Notation: $\mathfrak{B}$: a finite set of prime divisors B_i
$\mathfrak{D}_{\mathfrak{B}} := \{\mathbb{R}\text{-weil divisors } B \mid \operatorname{Supp} B = \sum_{B_i \in \mathfrak{B}} B_i\}$ ($\mathbb{R}$-vector space; B_i prime)
$\mathfrak{I}_{\mathfrak{B}} := \{\sum \beta_i B_i \in \mathfrak{D}_{\mathfrak{B}} \mid 0 \leq \beta_i \leq 1, \forall i\}$
unit cube in $\mathfrak{D}_{\mathfrak{B}}$

Lemma 1 (X, B) log pair B is $\mathbb{R}$-boundary.
Assume $K+B$ is n-complemented.
Then $\exists$ constant $\varepsilon = \varepsilon(X, B, n) > 0$ s.t. $K+B'$ is also n-complemented for any $\mathbb{R}$-boundary $B' \in \mathfrak{D}_{\mathfrak{B}}$ with $\|B - B'\| < \varepsilon$ ($\|B\| = \max(|b_1|, \cdots, |b_r|)$ norm)

The existence of n-complements is an open condition in the space of all boundaries B with fixed Supp B

Prop.1 Fix a positive integer I. X FT variety, K_X $\mathbb{Q}$-Cartier
let $B_1 \cdots B_r$ be $\mathbb{Q}$-Cartier $\mathfrak{B} := \sum_{i=1}^{r} B_i$
Then for any boundary $B \in \mathfrak{I}_{\mathfrak{B}}$ s.t. $K+B$ is lc and $-(K+B)$ is nef, there is an n-complement of $K+B$ for some $n \leq \operatorname{Const}(X, \mathfrak{B})$ and $I \mid n$

(1) setup: $(X, B = \sum b_i B_i)$ klt log semi-Fano, $\dim = d$, FT, $\mathbb{Q}$-divisor.
Assume Main theorem 1 and 2 hold in dimension $d-1$
By this inductive hypothesis,
whenever $\mathcal{R} \subset [0,1]$ finite rational $\Rightarrow \varepsilon_{d-1}(\bar{\mathcal{R}}) > 0$
Take any $0 < \varepsilon' \leq \varepsilon_{d-1}(\bar{\mathcal{R}})$. Put $I := I(\mathcal{R})$

> Adjunction on divisors
> $\mathcal{R} \subset \mathbb{R}_{\geq 0}$
> $\bar{\mathcal{R}} := \{r_0 - m\sum_{i=1}^{s}(1-r_i) \mid r_0, \cdots, r_s \in \mathcal{R}, m \in \mathbb{Z}, m>0\} \cap \mathbb{R}_{\geq 0}$
> $\mathcal{N}_d(\mathcal{R}) := \{m \in \mathbb{Z}_{>0} \mid \exists \text{ log semi-Fano variety } (X,D) \text{ of dim } d \text{ satisfying (i) (ii) (iii)}\}$
> (i) X is FT and $D \in \Phi(\mathcal{R})$
> (ii) either (X, D) is klt or $K_X + D \equiv 0$
> (iii) K_X+D is m-complemented, $I(\mathcal{R}) \mid m$, and m is minimal under these conditions
> $N_d = N_d(\mathcal{R}) := \sup \mathcal{N}_d(\mathcal{R})$,
> $\varepsilon_d = \varepsilon_d(\mathcal{R}) := 1/(N_d+2)$
>
> $\mathcal{R}$ finite $\Rightarrow$ $\sup \mathcal{N}_d(\mathcal{R})$
> $\Downarrow$
> $N_d > 0$

(1.1) • (X, B) is ε'-lt $\Rightarrow B \in \Phi(\mathcal{R}) \cap [0, 1-\varepsilon']$
The multiplicities of B are contained in this finite set
$\Downarrow$ conjecture 1
(X, B) is bounded
Hence $(X, \operatorname{Supp} B)$ belongs to an algebraic family
assume the multiplicities of B are fixed.
Let $m := nI$, $K+B$ is m-complemented
$\Updownarrow$
$\exists \bar{B} \in |-K - \lfloor (m+1)B \rfloor|$ s.t. $(X, \frac{1}{m}(\lfloor (m+1)B \rfloor + \bar{B}))$ is lc
open in the deformation space of $(X, \operatorname{Supp} B)$
By Prop 1 above, Noetherian induction $\Rightarrow$ log divisor $K+B$ has a bounded nI-complement for some $n \leq C(d, \mathcal{R})$

• Now we only need to consider the case (X, B) is not ε'-lt.
(A) "Replace"
$(X, B) \longrightarrow$ log crepant $\mathbb{Q}$-factorial blow up of all divisors E of discrepancy $a(E, X, B) \leq -1+\varepsilon'$
$B \in \Phi(\mathcal{R}) \longrightarrow B \in \Phi(\mathcal{R}, \varepsilon') \cap \mathbb{Q}$ $\Phi(\mathcal{R}, \varepsilon') := \Phi(\mathcal{R}) \cup [1-\varepsilon', 1]$
New X is also FT
Now assume X $\mathbb{Q}$-factorial and $\operatorname{discr}(X, B) > -1+\varepsilon'$

• Take some $n_0 \gg 0$ sufficient large
$n_0 B$ is integral and $|-n_0(K+B)|$ is base point free
$\bar{B}$ general member
$\Theta := B + \frac{1}{n_0}\bar{B}$ we have $\operatorname{discr}(X, \Theta) = \operatorname{discr}(X, B)$ (Bertini's theorem)
- $K_X + \Theta$ is a klt $\mathbb{Q}$-complement of $K_X + B$
- $\operatorname{discr}(X, \Theta) \geq -1+\varepsilon'$
- $\Theta - B$ is supported in a movable divisor

$n_0(K_X +$

A new boundary $D := \sum d_i B_i$ (components of B) $d_i = \begin{cases} 1 & \text{if } b_i \geq 1-\varepsilon' \\ b_i & \text{otherwise} \end{cases}$

$\begin{cases} \operatorname{Supp} D = \operatorname{Supp} B \\ D \in \Phi(\mathcal{R}) \\ D > B \\ \lfloor D \rfloor \neq 0 \\ (X, D) \text{ is lc} \end{cases}$

Lemma: (the simplest case of the global to local statement)
Fix a finite set $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$. let $(X \ni 0, D)$ be the germ of a $\mathbb{Q}$-factorial klt d-dimensional singularity. $D \in \Phi(\mathcal{R}, \varepsilon_{d-1}(\bar{\mathcal{R}}))$
Then there is an n-complement of $K_X + D$. $n \in \mathcal{N}_{d-1}(\bar{\mathcal{R}})$

> near each point $P \in X$
> n-complement $K+B^+$ of $K+B$
> $\mathcal{N}_{d-1}(\bar{\mathcal{R}})$ $\mathcal{P}_n \supset \Phi(\mathcal{R}, \varepsilon)$
> $B^+ \geq B$
> nB^+ is integral for any component of $D-B$
> coefficient $B \geq \varepsilon' > \frac{1}{n+1}$
> $B^+ \geq D \Rightarrow (X, D)$ is lc near P

• (B) Run $-(K+D)$-MMP
X is FT
If X is not FT, $K+B \equiv 0$ and $-(K+D)$-MMP coincides $K+B-\delta(D-B)$-MMP for some small $\delta > 0$

- $B \in \Phi(\mathcal{R}, \varepsilon')$
- All birational transformation are $(K+\Theta)$-crepant $\Rightarrow$ $K+\Theta$ is klt $\Downarrow B \leq \Theta$ $K+B$ is klt
- (X, D) is lc
- X FT

preserved on each step

Claim 1 None of components of $\lfloor D \rfloor$ is contracted.
Proof
proof: let $\varphi: X \to \bar{X}$ be a $K+D$-positive extremal contraction, $E \subset X$ corresponding exceptional divisor
Assume $E \subset \lfloor D \rfloor$. let $\bar{D} := \varphi_* D$
K_X+D is φ-ample $\Rightarrow$ we can write $K_X + D = \varphi^*(K_{\bar{X}} + \bar{D}) - \alpha E$, $\alpha > 0$
$(\bar{X}, \bar{D})$ is lc $\Rightarrow -1 \leq a(E, \bar{X}, \bar{D}) = a(E, X, D) - \alpha = -1 - \alpha < -1$ contradiction

• (C) Reduction. a number of divisorial contractions and flips
$X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N = Y$,
we get a $\mathbb{Q}$-factorial model Y s.t.
either • $\exists$ $\varphi: Y \to Z$
a non-birational $K_Y + D_Y$-positive extremal contraction to a lower-dimensional variety Z
or • $-(K_Y + D_Y)$ is nef.
In first case, Z is a point i.e., $\rho(Y) = 1$ and $-(K_Y + B_Y)$ is nef

• Now we have Y: $\mathbb{Q}$-factorial, FT
Two boundaries $B_Y = \sum b_i B_i$, $D_Y = \sum d_i B_i$ s.t $\begin{cases} \operatorname{discr}(Y, B_Y) > -1+\varepsilon' \\ B_Y \in \Phi(\mathcal{R}, \varepsilon') \\ \wedge\!\wedge \\ D_Y \in \Phi(\mathcal{R}) \end{cases}$ $d_i > b_i \Leftrightarrow d_i = 1$ and $b_i \geq -1+\varepsilon'$

Two cases: One of the following two cases holds
• $\rho(Y) = 1$, $K_Y + D_Y$ is ample, (Y, B_Y) klt log semi-Fano variety.
or • (Y, D_Y) log semi-Fano $\lfloor D_Y \rfloor \neq 0$ (since $D > B$ if $K+B \equiv 0$, the case does not occur)
treated in latter section

outline of the proof of Main Theorem 1
sketch the idea in the proof of boundedness in
(1.1) $\Rightarrow$ we only need to consider (X, B) is not ε'-lt
Do construction (A) (B) (C)
In each steps of divisorial contractions and flips in (C)
we contract an extremal ray $(K+D)$-positive

> Prop 2 $f: Y \to X$ birational contraction.
> D is $\mathbb{R}$-divisor on Y s.t
> (i) $K_Y + D$ is nef over X
> (ii) $f_* D \in \mathcal{P}_n := \{1 \leq \alpha \leq 1, \lfloor (n+1)\alpha \rfloor \geq n\alpha\}$ boundary $\cap \mathbb{R}$
> $K_X + f_* D$ is n-complemented
> $\Downarrow$
> $K_Y + D$ is n-complemented

we can pull back n-complements of $K_Y + D_Y$ to original X. $n \in \mathcal{N}_d(\mathcal{R})$
$\rho(Y) = 1$, it possible that $K_Y + D_Y$ has no complements
In this case
Section 6 $\Rightarrow$ $b_i < 1 - c$, where $c > 0$ the multiplicities of B_Y are boundled from the above

Claim 1 $\Rightarrow$ divisorial contractions in (C) do not contract components of B with $b_i \geq 1 - \varepsilon'$
$\Downarrow$
B is also bounded from above.
preserved in each step $\rightarrow$ $\operatorname{discr}(X, B) > -1 + \varepsilon'$
Conjecture 1
$\Rightarrow$ $(X, \operatorname{Supp} B)$ belong to an algebraic family

Noetherian induction $\Rightarrow$ assume $(X, \operatorname{Supp} B)$ is fixed

By Prop 1. (X, B) has bounded complements ∎

• Proof of Main theorem 2 in the case when (X, B) is not klt
Lemma 2. $(X, B = \sum b_i B_i)$ 0-pair of dimension d s.t. $B \in \Phi(\mathcal{R}, \varepsilon')$, where $\varepsilon' := \varepsilon_{d-1}(\bar{\mathcal{R}})$.
Assume the LMMP in dim d
Assume either (i) (X, B) is not klt and Main theorem 1 and 2 hold in dimension $d-1$
or (ii) Main Theorem 2. holds in dim d
Then $\exists$ $\lambda := \lambda(d, \mathcal{R}) > 0$ s.t either $b_i = 1$ or $b_i \leq 1 - \lambda$ for all b_i

- Corollary of lemma 2 $a(E, X, B) = 1$ or $a(E, X, B) \leq 1 - \lambda$ for any divisor on X (exceptional or not)

• Proof of Main theorem 2 in the case when (X, B) is not klt
(X, B) 0-pair not klt
X FT, $B \in \Phi(\mathcal{R})$
Replace $(X, B) \longrightarrow$ its $\mathbb{Q}$-factorial dlt modification $\Rightarrow$ X is klt, $\lfloor B \rfloor \neq 0$, $B \in \Phi(\mathcal{R})$

let $0 < \lambda < \lambda(d, \mathcal{R})$
If X is not λ-lt $\Rightarrow$ (Coro of Lemma 2) $a(E, X, 0) < -1 + \lambda$ for each exceptional divisor E
$a(E, X, B) = 1$
As in (A) replace $(X, B) \longrightarrow$ blow up of all such divisors E
Then we have (X, B) 0-pair, X λ-lt, $B \in \Phi(\mathcal{R})$ $\lfloor B \rfloor \neq 0$

- Run K-MMP: $X \dashrightarrow X'$ B' is biration transform of B
$B \neq 0$ $\Rightarrow$ $\exists$ $X' \to Z'$ $\dim Z' < \dim X'$
K-negative Fano fibration
By our construction (X', B') not klt 0-pair, X' λ-lt, $B' \in \Phi(\mathcal{R})$
Prop 2 $\Rightarrow$ we can pull back n-complements from X' to X if $I(\mathcal{R}) \mid n$
Case 1: Z' is a point $\Rightarrow$ $\begin{cases} \rho(X') = 1 \\ X' \text{ is klt Fano} \end{cases}$
(1.1) $\Rightarrow$ $(X', \operatorname{supp} B')$ belong to an algebraic family
Prop 1, Noetherian induction $\Rightarrow$ log divisor $K+B$ has bounded nI-complement for some $n \leq C(d, \mathcal{R})$
Case 2. $\dim Z' > 0$ treated by results in latter sections. ∎